

Dain's invariant on non-time symmetric initial data sets

J.A. Valiente Kroon ^{*,1} and J.L. Williams ^{†,1}

¹*School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom.*

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Abstract

We extend Dain's construction of a geometric invariant characterising static initial data sets for the vacuum Einstein field equations to situations with a non-vanishing extrinsic curvature. This invariant gives a measure of how much an initial data set with non-vanishing ADM 4-momentum deviates from stationarity. In particular, it vanishes if and only if the initial data set is stationary. Thus, the invariant provides a quantification of the amount of gravitational radiation contained in the initial data set.

1 Introduction

The Cauchy problem for the Einstein field equations is a cornerstone of mathematical Relativity. Indeed, the proper rigorous formulation of many of the outstanding problems in mathematical Relativity, such as stability of certain special solutions, is made within the framework of the Cauchy problem. Accordingly, one of the challenges in the construction of the spacetimes by means of the Cauchy problem is to provide physically relevant initial data for the evolution equations of General Relativity.

As it is well known, initial data for the evolution equations of General Relativity cannot be freely specified —in order to obtain a proper solution to the Einstein field equations, the initial data set has to satisfy the so-called Einstein constraint equations. Solutions to the Einstein constraint equations have been studied extensively —see e.g. [4]. Of particular relevance in this respect is the question of under which conditions an initial data set gives rise to a spacetime development possessing Killing symmetries —this question first arose in the context of linearisation stability, see [19]. These conditions are encoded in the so-called *Killing initial data (KID) equations* —see e.g. [10, 6] for a discussion of the basic properties of these equations; see also [12]. The KID equations constitute a system of overdetermined equations for a scalar and a vector on the initial hypersurface. The existence of a solution to these equations is equivalent to the existence of a Killing vector in the development of the initial data set. The KID equations have a deep connection with the *Arnowit-Deser-Misner (ADM) evolution equations*: the evolution equations can be described as a flow generated by the adjoint linearised constraint map, $D\Phi^*$ (see below) —see e.g. [14] for further details.

In many applications of both physical and mathematical interest it is important to have a way of quantifying how much a given initial data set deviates from stationarity. Ideally, one would like to do this in coordinate-independent manner. One approach to this problem was proposed in [13], in which the notion of an *approximate Killing vector*, as a solution to a fourth-order linear elliptic system arising from the KID equations, was introduced. The so-called *approximate Killing vector equation* has the property that its kernel contains that of the KID equations. The

*E-mail address: j.a.valiente-kroon@qmul.ac.uk

†E-mail address: j.l.williams@qmul.ac.uk

analysis in [13] was restricted to the case of time symmetric asymptotically Euclidean initial data sets. In particular, it was shown that the kernel of the approximate Killing vector equation is non-trivial, and moreover that, given suitable assumptions on the asymptotics of the initial data set, the solution (termed the *approximate Killing vector*) is unique up to constant rescalings.

It is of interest to mention that the general strategy adopted in [13] has found applicability in the analysis of spacetimes admitting a Killing spinor —see [15]. These ideas have been used, in turn, to obtain an invariant characterising initial data sets for the Kerr spacetime, see [1, 2], and for the Kerr-Newman spacetime, see [11].

The purpose of this article is to extend Dain’s result in [13] to the non-time symmetric case. Moreover, we analyse in some detail conformally flat initial data sets as way of obtaining some further insight into Dain’s construction. Our main result is Theorem 1 which shows that the approximate Killing vector equation can be solved with the required asymptotic conditions for a large class of asymptotically Euclidean initial data sets.

Overview of the article

This article is structured as follows: Section 2 provides a discussion of the basic properties of the approximate Killing vector equation as introduced by Dain. In particular, Subsection 2.1 provides a discussion of the relation between the Einstein constraint equations and the so-called Killing Initial Data (KID) equations; Subsection 2.2 provides a detailed discussion of the approximate Killing vector equation in the non-time symmetric setting; Subsection 2.3 introduces some useful identities which will be used throughout. Section 3 analyses the solvability of the approximate Killing equation on asymptotically Euclidean manifolds: in Subsection 3.1 some basic background on weighted Sobolev spaces is given; Subsection 3.2 provides a discussion of our main asymptotic decay assumptions and of the asymptotic behaviour of solutions to the KID equations; Subsection 3.3 briefly reviews the basic methods to analyse the existence of solutions to elliptic equations on asymptotically Euclidean manifolds; Subsection 3.4 contains our main existence results. Finally, Section 4 contains a further discussion of the geometric invariant obtained from Dain’s construction with particular emphasis to the case of conformally flat initial data sets.

Notation and conventions

We use Penrose’s abstract index notation throughout so that i, j, k, \dots denote abstract 3-dimensional tensorial indices. The Greek indices $\alpha, \beta, \gamma, \dots$ denote 3-dimensional coordinate indices. Riemannian 3-dimensional metrics are assumed to have signature $(+++)$. Our conventions for the curvature are fixed by

$$D_i D_j v^k - D_j D_i v^k = r^k{}_{lij} v^l.$$

The Ricci r_{lj} tensor is obtained from the Riemann tensor via the relation

$$r_{lj} \equiv r^i{}_{lij}.$$

2 The approximate Killing vector equation

In this section we introduce the basic objects of our analysis: the vacuum Einstein constraint equations, the Killing initial data equations and the approximate Killing initial data equations.

2.1 The Einstein constraints and the KID equations

In this article we will study properties of initial data sets for the vacuum Einstein field equations —that is, triples $(\mathcal{S}, h_{ij}, K_{ij})$ where \mathcal{S} is a 3-dimensional manifold, h_{ij} is a Riemannian metric on \mathcal{S} and K_{ij} is a symmetric rank 2 tensor satisfying the vacuum Einstein constraint equations

$$r + K^2 - K_{ij} K^{ij} = 0, \tag{1a}$$

$$D^j K_{ij} - D_i K = 0. \tag{1b}$$

Following the standard conventions we refer to equations (1a) and (1b) as the *Hamiltonian* and *momentum* constraints, respectively. In the above expressions D_i denotes the Levi-Civita connection of the metric h_{ij} and r is the associated Ricci scalar. Furthermore, $K \equiv K_{ij}h^{ij}$.

In the following we will be particularly interested on initial data sets $(\mathcal{S}, h_{ij}, K_{ij})$ whose development has a Killing vector. The conditions for this to be case are identified in the following:

Proposition 1. *Let $(\mathcal{S}, h_{ij}, K_{ij})$ denote an initial data set for the vacuum Einstein field equations. If there exists a scalar field N and a vector field Y^i over \mathcal{S} satisfying the equations*

$$L_{ij} \equiv NK_{ij} + D_{(i}Y_{j)} = 0, \quad (2a)$$

$$M_{ij} \equiv Y^k D_k K_{ij} + D_i Y^k K_{kj} + D_j Y^k K_{ik} + D_i D_j N - N(r_{ij} + K K_{ij} - 2K_{ik}K^k_j) = 0, \quad (2b)$$

then the development of the initial data is endowed with a Killing vector.

A proof of this result can be found in e.g. [10] —see also [6].

Remark 1. The pair (N, Y^i) is called a *Killing initial data set (KID)* and equations (2a)-(2b) are known as the *KID equations*.

It is interesting to note that Killing initial data for conformally rescaled vacuum spacetimes has been analysed in [20, 21], with applications to the characterisation of Kerr-de Sitter-like spacetimes in [17].

2.2 Basic properties of the approximate Killing vector equation

In the following, denote by \mathcal{M}_2 , \mathcal{S}_2 , \mathcal{X} and \mathcal{C} the spaces of Riemannian metrics, symmetric 2-tensors, vectors and scalar functions on the 3-dimensional manifold \mathcal{S} , respectively. It is convenient to write the Einstein constraint equations (1a) and (1b) in terms of a map (*the constraint operator*)

$$\Phi : \mathcal{M}_2 \times \mathcal{S}_2 \rightarrow \mathcal{C} \times \mathcal{X}$$

such that for $h_{ij} \in \mathcal{M}_2$, $K_{ij} \in \mathcal{S}_2$ one has

$$\Phi \left(\begin{array}{c} h_{ij} \\ K_{ij} \end{array} \right) \equiv \left(\begin{array}{c} r + K^2 - K_{ij}K^{ij} \\ -D^j K_{ij} + D_i K \end{array} \right).$$

In terms of the latter, the constraints (1a) and (1b) take the form

$$\Phi \left(\begin{array}{c} h_{ij} \\ K_{ij} \end{array} \right) = 0.$$

The *linearisation of the constraint operator* Φ , $D\Phi : \mathcal{S}_2 \times \mathcal{S}_2 \rightarrow \mathcal{C} \times \mathcal{X}$, evaluated at (h_{ij}, K_{ij}) can be found to be given by

$$D\Phi \left(\begin{array}{c} \gamma_{ij} \\ Q_{ij} \end{array} \right) = \left(\begin{array}{c} D^i D^j \gamma_{ij} - r_{ij} \gamma^{ij} - \Delta_h \gamma + H \\ -D^j Q_{ij} + D_i Q - F_i \end{array} \right)$$

where $\gamma \equiv h^{ij} \gamma_{ij}$, $Q \equiv h^{ij} Q_{ij}$ and

$$H \equiv 2(KQ - K^{ij}Q_{ij}) + 2(K^{ki}K^j_k - K K^{ij})\gamma_{ij},$$

$$F_i \equiv (D_i K^{kj} - D^k K^j_i)\gamma_{jk} - (K^k_i D^j - \frac{1}{2}K^{kj}D_i)\gamma_{jk} + \frac{1}{2}K^k_i D_k \gamma,$$

while $\Delta_h \equiv h^{ij}D_i D_j$ is the Laplacian of the metric h_{ij} . Moreover, using integration by parts, the formal adjoint of the linearised constraint operator, $D\Phi^* : \mathcal{C} \times \mathcal{X} \rightarrow \mathcal{S}_2 \times \mathcal{S}_2$, can be seen to be given by

$$D\Phi^* \left(\begin{array}{c} X \\ X_i \end{array} \right) = \left(\begin{array}{c} D_i D_j X - X r_{ij} - \Delta_h X h_{ij} + H_{ij} \\ D_{(i} X_{j)} - D^k X_k h_{ij} + F_{ij} \end{array} \right)$$

where

$$\begin{aligned} H_{ij} &\equiv 2X(K^k{}_i K_{jk} - K K_{ij}) - K_{k(i} D_{j)} X^k + \frac{1}{2} K_{ij} D_k X^k \\ &\quad + \frac{1}{2} K_{kl} D^k X^l h_{ij} - \frac{1}{2} X^k D_k K_{ij} + \frac{1}{2} X^k D_k K h_{ij}, \\ F_{ij} &\equiv 2X(K h_{ij} - K_{ij}). \end{aligned}$$

Note that in the case of time-symmetric data, $H = F_i = H_{ij} = F_{ij} \equiv 0$, and the above expressions for $D\Phi$ and $D\Phi^*$ thereby reduce to those given in [13].

Remark 2. A calculation shows that $D\Phi^* = 0$ is equivalent to the KID equations (2a)-(2b). Indeed, one has that

$$D\Phi^* \begin{pmatrix} N \\ -2Y_i \end{pmatrix} = \begin{pmatrix} M_{ij} - M_k{}^k h_{ij} - \frac{1}{2} K_{kl} L^{kl} h_{ij} + \frac{1}{2} K_{ij} L_k{}^k \\ L_{ij} - L_k{}^k h_{ij} \end{pmatrix}$$

from which we see that $D\Phi^*(N, -2Y_i) = 0$ if and only if $L_{ij} = M_{ij} = 0$ —i.e. if and only if (N, Y^i) satisfy the KID equations.

Now, let $\mathcal{S}_{1,2}$ denote the space of covariant rank-3 tensors which are symmetric in the last two indices. Following Dain [13], we consider an operator $\mathcal{P} : \mathcal{S}_2 \times \mathcal{S}_{1,2} \rightarrow \mathcal{C} \times \mathcal{X}$ such that

$$\mathcal{P} \begin{pmatrix} \gamma_{ij} \\ q_{kij} \end{pmatrix} \equiv D\Phi \begin{pmatrix} \gamma_{ij} \\ -D^k q_{kij} \end{pmatrix}$$

with formal adjoint, $\mathcal{P}^* : \mathcal{C} \times \mathcal{X} \rightarrow \mathcal{S}_2 \times \mathcal{S}_{1,2}$, given by

$$\mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & D_k \end{pmatrix} \cdot D\Phi^* \begin{pmatrix} X \\ X_i \end{pmatrix} = \begin{pmatrix} D_i D_j X - X r_{ij} - \Delta_h X h_{ij} + H_{ij} \\ D_k (D_{(i} X_{j)} - D^l X_l h_{ij} + F_{ij}) \end{pmatrix}.$$

Further, we consider the composition $\mathcal{P} \circ \mathcal{P}^* : \mathcal{C} \times \mathcal{X} \rightarrow \mathcal{C} \times \mathcal{X}$, given by

$$\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} \equiv \begin{pmatrix} 2\Delta_h \Delta_h X - r^{ij} D_i D_j X + 2r \Delta_h X + \frac{3}{2} D^i r D_i X + (\frac{1}{2} \Delta_h r + r_{ij} r^{ij}) X \\ \quad + D^i D^j H_{ij} - \Delta_h H_k{}^k - r^{ij} H_{ij} + \bar{H} \\ D^j \Delta_h D_{(i} X_{j)} + D_i \Delta_h D^k X_k + D^j \Delta_h F_{ij} - D_i \Delta_h F_k{}^k - \bar{F}_i \end{pmatrix}$$

where

$$\begin{aligned} \bar{H} &\equiv 2(K\bar{Q} - K^{ij}\bar{Q}_{ij}) + 2(K^{ki}K^j{}_k - K K^{ij})\bar{\gamma}_{ij}, \\ \bar{F}_i &\equiv (D_i K^{kj} - D^k K^j{}_i) \bar{\gamma}_{jk} - (K^k{}_i D^j - \frac{1}{2} K^{kj} D_i) \bar{\gamma}_{jk} + \frac{1}{2} K^k{}_i D_k \bar{\gamma} \\ \bar{\gamma}_{ij} &\equiv D_i D_j X - X r_{ij} - \Delta_h X h_{ij} + H_{ij} \\ \bar{Q}_{ij} &\equiv -\Delta_h (D_{(i} X_{j)} - D^k X_k h_{ij} + F_{ij}) \end{aligned}$$

and F_{ij} , H_{ij} as above. One has the following:

Lemma 1. *The operator $\mathcal{P} \circ \mathcal{P}^* : \mathcal{C} \times \mathcal{X} \rightarrow \mathcal{C} \times \mathcal{X}$ as defined above is a self-adjoint fourth order elliptic operator.*

Proof. The self-adjointness follows from the definition as the operator is obtained by the composition of an operator and its formal adjoint. To verify the ellipticity of the operator we notice that the symbol is given by

$$\sigma_\xi \begin{pmatrix} X \\ X_i \end{pmatrix} = \begin{pmatrix} 2|\xi|^2 X \\ \xi^j |\xi|^2 \xi_{(i} X_{j)} + \xi_i |\xi|^2 \xi_j X^j \end{pmatrix}$$

for ξ_i a covector and $|\xi|^2 \equiv \delta_{ij} \xi^i \xi^j$. Clearly, the first component is an isomorphism if $|\xi|^2 \neq 0$. For the second component, contract first with ξ^i to get $2|\xi|^4 \xi^j X_j = 0$ for X_i in the kernel, which implies $\xi^j X_j = 0$. Substituting back into the symbol, one obtains that $|\xi|^4 X_i = 0$. So, for $|\xi|^2 \neq 0$, the symbol is injective. Clearly the codomain has the same dimension as the domain, and therefore σ_ξ is an isomorphism for $|\xi|^2 \neq 0$ —i.e. $\mathcal{P} \circ \mathcal{P}^*$ is fourth-order elliptic operator. \square

The previous discussion suggests the following:

Definition 1. *The equation*

$$\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} = 0 \quad (3)$$

will be called the approximate Killing initial data (KID) equation and a solution (X, X^i) thereof an approximate Killing initial data set —or approximate KID for brevity.

Remark 3. As pointed out in [13], the equation $\mathcal{P} \circ \mathcal{P}^*(X, X_i) = 0$ is the Euler–Lagrange equation of the action

$$\int_{\mathcal{U}} \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} \cdot \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} d\mu$$

Note that, had we used the operator $D\Phi^*$ rather than \mathcal{P}^* , then the pointwise norm defined by the integrand would contain terms of inconsistent physical dimension: $[X] = L^{-2}$, $[X_i] = 1$, and so for instance $[(D_i D_j X)(D^i D^j X)] = L^{-6}$, while $[D_{(i} X_{j)} D^{(i} X^{j)}] = L^{-4}$.

2.3 Integration by parts identities

The expressions in the previous subsection and several of our arguments in latter parts are based on integration by parts. For quick reference, in this subsection we provide the integral expressions relating the operators \mathcal{P} and \mathcal{P}^* including boundary terms.

Let $\mathcal{U} \subset \mathcal{S}$ denote a compact set with boundary $\partial\mathcal{U}$. Recall that by definition

$$\begin{aligned} \int_{\mathcal{U}} \begin{pmatrix} X \\ X^i \end{pmatrix} \cdot \mathcal{P} \begin{pmatrix} \gamma_{ij} \\ q_{kij} \end{pmatrix} d\mu &= \int_{\mathcal{U}} \begin{pmatrix} X \\ X^i \end{pmatrix} \cdot D\Phi \begin{pmatrix} \gamma_{ij} \\ -D^k q_{kij} \end{pmatrix} d\mu \\ &= \int_{\mathcal{U}} \begin{pmatrix} X \\ X^i \end{pmatrix} \cdot \begin{pmatrix} D^i D^j \gamma_{ij} - r_{ij} \gamma^{ij} - \Delta_{\mathbf{h}} \gamma + H \\ D^j D^k q_{kij} - D_i D^k q_{kj}{}^j - F_i \end{pmatrix} d\mu \\ &= \int_{\mathcal{U}} X (D^i D^j \gamma_{ij} - r_{ij} \gamma^{ij} - \Delta_{\mathbf{h}} \gamma + H) d\mu \\ &\quad + \int_{\mathcal{U}} X^i (D^j D^k q_{kij} - D_i D^k q_{kj}{}^j - F_i) d\mu \\ &= J_1 + J_2. \end{aligned}$$

We now proceed to use integration by parts on J_1 and J_2 . A lengthy computation shows that

$$\begin{aligned} J_1 &\equiv \int_{\mathcal{U}} X (D^i D^j \gamma_{ij} - r_{ij} \gamma^{ij} - \Delta_{\mathbf{h}} \gamma + H) d\mu \\ &= \int_{\mathcal{U}} \gamma_{ij} (D^j D^i X - h^{ij} \Delta_{\mathbf{h}} X - X r^{ij} + 2(K^{ki} K^j{}_k - K K^{ij})) d\mu \\ &\quad + \int_{\mathcal{U}} 2q_{kij} (h^{ij} X D^k K + h^{ij} K D^k X - X D^k K^{ij} - K^{ij} D^k X) d\mu \\ &\quad + \oint_{\partial\mathcal{U}} n^k (\mathcal{A}_k + \mathcal{B}_k) dS \end{aligned}$$

where the boundary integrands are given by

$$\begin{aligned} \mathcal{A}_k &\equiv X D^j \gamma_{jk} - D^j X \gamma_{jk} - D_k X \gamma - X D_k \gamma, \\ \mathcal{B}_k &\equiv 2(K^{ij} q_{kij} - K q_{kj}{}^j) X. \end{aligned}$$

Similarly, one finds that

$$\begin{aligned}
J_2 &\equiv \int_{\mathcal{U}} X^i (D^j D^k q_{kij} - D_i D^k q_{kj}{}^j - F_i) d\mu \\
&= \int_{\mathcal{U}} q_{kij} (D^k D^j X^i + D^k D_l h^{ij}) d\mu \\
&\quad - \int_{\mathcal{U}} \gamma_{jk} ((D_i K^{kj} - D^k K^j{}_i) X^i + D^j (X^i K_i{}^k) - \tfrac{1}{2} D_i (X^i K^{kj}) - \tfrac{1}{2} h^{jk} D_i (X^l K_l{}^i)) d\mu \\
&\quad + \oint_{\partial\mathcal{U}} n^k (\mathcal{C}_k + \mathcal{D}_k) dS
\end{aligned}$$

where the boundary integrands are given by

$$\begin{aligned}
\mathcal{C}_k &\equiv X^i D^l q_{lik} - D^j X^i q_{kij} + D_i X^i q_{kj}{}^j - X_i D^l q_{lj}{}^j, \\
\mathcal{D}_k &\equiv X^i K_i{}^l \gamma_{kl} - \tfrac{1}{2} X_k K^{lj} \gamma_{jl} - \tfrac{1}{2} X^i K_{ik} \gamma.
\end{aligned}$$

Putting everything together and after some further manipulations one finds the identity

$$\int_{\mathcal{U}} \begin{pmatrix} X \\ X^i \end{pmatrix} \cdot \mathcal{P} \begin{pmatrix} \gamma_{ij} \\ q_{kij} \end{pmatrix} d\mu = \int_{\mathcal{U}} \begin{pmatrix} \gamma^{ij} \\ q^{kij} \end{pmatrix} \cdot \mathcal{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} + \oint_{\partial\mathcal{U}} n^k (\mathcal{A}_k + \mathcal{B}_k + \mathcal{C}_k + \mathcal{D}_k) dS. \quad (4)$$

3 The approximate Killing vector equation on asymptotically Euclidean manifolds

In this section we study the solvability of the approximate KID equation on asymptotically Euclidean manifolds. The standard methods to study elliptic equations on this type of manifolds employ so-called *weighted Sobolev spaces* —thus, we start by briefly reviewing our basic technical tools in Section 3.1. The key assumption on the class of initial data sets to be considered are discussed in 3.2. The existence results for the approximate KID equation are given in Subsection 3.

3.1 Weighted Sobolev spaces

In order to discuss the decay of the various tensor fields in the 3-manifold \mathcal{S} we need to make use of *weighted Sobolev spaces* —see e.g. [8, 16, 9, 3]. Given an arbitrary point $p \in \mathcal{S}$ one defines for $x \in \mathcal{S}$

$$\sigma(x) \equiv (1 + d(p, x)^2)^{1/2}$$

where $d(p, x)$ denotes the Riemannian distance on \mathcal{S} . The function σ is used to define the weighted L^2 -norm

$$\|u\|_{\delta} \equiv \left(\int_{\mathcal{S}} |u|^2 \sigma^{-2\delta-3} d^3x \right)^{1/2}, \quad \delta \in \mathbb{R}.$$

In particular, if $\delta = -3/2$ one recovers the usual L^2 -norm. Different choices of origin give rise to equivalent weighted norms.

Remark 4. In the above and in the rest of the article, we follow Bartnik's conventions [3] to denote the weighted Sobolev spaces and norms. Different choices of the point p give rise to equivalent weighted norms —see e.g. [3, 9]. Thus, in a slight abuse of notation, we denote all these equivalent norms with the same symbol.

The fall-off behaviour of the various fields will be expressed in terms of weighted Sobolev spaces H_{δ}^s consisting of functions for which

$$\|u\|_{s, \delta} \equiv \sum_{0 \leq |\alpha| \leq s} \|D^{\alpha} u\|_{\delta-|\alpha|} < \infty,$$

where s is a non-negative integer, and where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multiindex, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. One says that $u \in H_s^\infty$ if $u \in H_\delta^s$ for all s . We will say that a tensor belongs to a given function space if its norm does.

In the following given some coordinates $x = (x^\alpha)$, let $|x|^2 \equiv \delta_{\alpha\beta} x^\alpha x^\beta$. We will make repeated use of the following result¹:

Lemma 2. *Let $u \in H_\delta^\infty$. Then u is smooth (i.e. C^∞) over \mathcal{S} and has a fall-off at infinity such that*

$$D^l u = o(|x|^{\delta-|l|}).$$

The proof can be found in [3] —see also Section 6.1 in [2]. The following *Multiplication Lemma* has been proven in [2]:

Lemma 3. *Let $u = o_\infty(|x|^{\delta_1})$, $v = o_\infty(|x|^{\delta_2})$ and $w = O(|x|^\gamma)$. Then*

$$uv = o_\infty(|x|^{\delta_1+\delta_2}), \quad uw = o_\infty(|x|^{\delta_1+\gamma}).$$

Remark 5. This lemma can be readily extended to tensor fields.

3.2 Decay assumptions

In what follows we will consider initial data sets $(\mathcal{S}, h_{ij}, K_{ij})$ for the vacuum Einstein field equations possessing, in principle, several *asymptotically Euclidean ends*. Thus, we assume there exists a compact set \mathcal{B} such that

$$\mathcal{S} \setminus \mathcal{B} = \sum_{k=1}^n \mathcal{S}_{(k)}$$

where $\mathcal{S}_{(k)}$, $k = 1, \dots, n$, are open sets diffeomorphic to the complement of a closed ball on \mathbb{R}^3 . Each set $\mathcal{S}_{(k)}$ is called an *asymptotic end*. On each of these ends one can introduce (non-unique) *asymptotically Cartesian* coordinates $x = (x^\alpha)$. Our basic decay assumptions for the fields h_{ij} and K_{ij} are expressed in terms of these coordinates:

Assumption 1 (Decay Assumptions). On each asymptotically Euclidean end one has

$$\begin{aligned} h_{\alpha\beta} - \delta_{\alpha\beta} &= o_\infty(|x|^{-1/2}), \\ K_{\alpha\beta} &= o_\infty(|x|^{-3/2}). \end{aligned}$$

We assume further that the decay is such that the ADM 4-momentum is non-vanishing.

The following definition will prove useful:

Definition 2. *An asymptotically Euclidean initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ satisfying the Decay Assumptions 1 is said to be stationary if there exists non-trivial $(N, N^i) \in H_{1/2}^2$ such that*

$$\mathcal{P}^* \begin{pmatrix} N \\ N^i \end{pmatrix} = 0, \tag{5}$$

Remark 6. As it is to be expected, a stationary initial data set (in the sense of Definition 2) admits a KID. To see this, observe that equation (5) implies that

$$D_i D_j N - N r_{ij} - \Delta_{\mathbf{h}} N h_{ij} + H_{ij} = 0, \tag{6a}$$

$$D_k (D_{(i} N_{j)} - D^l N_l h_{ij} + F_{ij}) = 0. \tag{6b}$$

Direct inspection shows that, if one assumes $(N, N^i) \in H_{1/2}^2$ in addition to decay Assumption 1, then

$$D_{(i} N_{j)} - D^l N_l h_{ij} + F_{ij} = o(|x|^{-1/2})$$

¹Recall that $f(x) = o(|x|^\alpha)$ if $f(x)/|x|^\alpha \rightarrow 0$ as $|x| \rightarrow 0$. If $\partial^n f(x) = o(|x|^{\alpha-n})$ for each non-negative integer, then we write $f(x) = o_\infty(|x|^\alpha)$.

—i.e. the tensor field $D_{(i}N_{j)} - D^l N_l h_{ij} + F_{ij}$ vanishes at infinity. Since it is also covariantly constant as a consequence of (6b), it follows then that it must vanish identically —i.e.

$$D_{(i}N_{j)} - D^l N_l h_{ij} + F_{ij} = 0.$$

Combining this observation with (6a), we see that $D\Phi^*(N, N^i) = 0$ and hence $(N, -\frac{1}{2}N^i)$ solves the KID equations (2a)-(2b) —see Remark 2. Finally, we observe that the behaviour $(N, -\frac{1}{2}N^i) = o(|x|^{1/2})$ for a KID is only consistent with *translational Killing vector fields* —i.e. Killing vectors which to leading order look like a (timelike, spatial or null) translation in the Minkowski spacetime. Now, the only type of translational Killing vector a spacetime with non-vanishing ADM 4-momentum can admit is one which is timelike and bounded at infinity —i.e. a stationary Killing vector, see Section III in [5]. Clearly the reverse is also true: if an initial data set admits a stationary Killing vector, then the data is stationary in the sense of Definition 2. It should be stressed that *our definition of stationary initial data sets excludes initial data sets for the Minkowski spacetime* as these necessarily have a vanishing ADM 4-momentum. The condition on the ADM 4-momentum in Definition 2 arises from the need to single out the stationary Killing vector field from among the collection of translational Killing vectors.

The asymptotic behaviour of solutions to the KID equations has been studied in [5] from where we adapt the following result:

Proposition 2. *Let $(\mathcal{S}, h_{ij}, K_{ij})$ denote a smooth vacuum initial data set satisfying the Decay Assumptions 1. Moreover, let N, Y^i be, respectively, a smooth scalar field and a vector field over \mathcal{S} satisfying the KID equations. Then, there exists a constant tensor with components $\mathfrak{L}_{\mu\nu} = \mathfrak{L}_{[\mu\nu]}$ such that*

$$N - \mathfrak{L}_{0\alpha}x^\alpha = o_\infty(|x|^{1/2}), \quad Y^\alpha - \mathfrak{L}_{\alpha\beta}x^\beta = o_\infty(|x|^{1/2}).$$

If $\mathfrak{L}_{\mu\nu} = 0$, then there exists a constant vector with components \mathfrak{A}^μ such that

$$N - \mathfrak{A}^0 = o_\infty(|x|^{-1/2}), \quad Y^\alpha - \mathfrak{A}^\alpha = o_\infty(|x|^{-1/2}).$$

Finally, if $\mathfrak{L}_{\mu\nu} = \mathfrak{A}^\mu = 0$, then $N = 0$ and $Y^i = 0$.

3.3 Basic results of the theory of elliptic equations on asymptotically Euclidean manifolds

In view of the Decay Assumptions 1, the approximate KID equation (3) can be written, in local coordinates, in the form

$$\mathcal{L}\mathbf{u} \equiv (\mathbf{A}^{\alpha\beta\gamma\delta} + \mathbf{a}^{\alpha\beta\gamma\delta}) \cdot \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \mathbf{u} + \mathbf{a}^{\alpha\beta\gamma} \cdot \partial_\alpha \partial_\beta \partial_\gamma \mathbf{u} + \mathbf{a}^{\alpha\beta} \cdot \partial_\alpha \partial_\beta \mathbf{u} + \mathbf{a}^\alpha \cdot \partial_\alpha \mathbf{u} + \mathbf{a} \cdot \mathbf{u} = 0,$$

where $\mathbf{u} : \mathcal{S} \rightarrow \mathbb{R}^4$ is a vector-valued function over \mathcal{S} , $\mathbf{A}^{\alpha\beta\gamma\delta}$ denote constant matrices, while $\mathbf{a}^{\alpha\beta\gamma\delta}$, $\mathbf{a}^{\alpha\beta\gamma}$, $\mathbf{a}^{\alpha\beta}$, \mathbf{a}^α and \mathbf{a} denote smooth matrix-valued functions of the coordinates $x = (x^\alpha)$.

The operator \mathcal{L} is said to be *asymptotically homogeneous* if

$$\mathbf{a}^{\alpha\beta\gamma\delta} \in H_\tau^\infty, \quad \mathbf{a}^{\alpha\beta\gamma} \in H_{\tau-1}^\infty, \quad \mathbf{a}^{\alpha\beta} \in H_{\tau-2}^\infty, \quad \mathbf{a}^\alpha \in H_{\tau-3}^\infty, \quad \mathbf{a} \in H_{\tau-4}^\infty,$$

for some $\tau < 0$ —see e.g. [8, 16].

Remark 7. Direct inspection using the Decay Assumptions 1 imply that \mathcal{L} is asymptotically homogeneous with $\tau = -1/2$. This is the standard assumption when working with weighted Sobolev spaces.

In the following we will make use of the following version of the *Fredholm alternative* for fourth-order asymptotically homogeneous operators on asymptotically Euclidean manifolds —see [8]:

Proposition 3. *Let \mathcal{L} be an asymptotically homogeneous elliptic operator of order 4 with smooth coefficients. Given δ not a negative integer, the equation*

$$\mathcal{L}\mathbf{u} = \mathbf{f}, \quad \mathbf{f} \in H_{\delta-4}^0$$

has a solution $\mathbf{u} \in H_{\delta}^4$ if and only if

$$\int_S \mathbf{f} \cdot \mathbf{v} \, d\mu = 0$$

for all \mathbf{v} satisfying

$$\mathcal{L}^*\mathbf{v} = 0, \quad \mathbf{v} \in H_{1-\delta}^0,$$

where \mathcal{L}^ denotes the formal adjoint of \mathcal{L} .*

Finally, to assert the regularity of solutions we need the following elliptic estimate —see Theorem 6.3. of [8]:

Proposition 4. *Let \mathcal{L} be an asymptotically homogeneous elliptic operator of order 4 with smooth coefficients. Then for any $\delta \in \mathbb{R}$ and any $s \geq 4$, there exists a constant C such that for every $\mathbf{v} \in H_{loc}^s \cap H_{\delta}^0$, the following inequality holds:*

$$\|\mathbf{v}\|_{H_{\delta}^s} \leq C \left(\|\mathcal{L}\mathbf{v}\|_{H_{\delta-2}^{s-4}} + \|\mathbf{v}\|_{H_{\delta}^{s-4}} \right).$$

In the above proposition H_{loc}^s denotes the local Sobolev space —that is, $\mathbf{v} \in H_{loc}^s$ if for an arbitrary smooth function ϕ with compact support, $\phi\mathbf{v} \in H^s$.

Remark 8. If \mathcal{L} has smooth coefficients and $\mathcal{L}\mathbf{v} = 0$, then it follows that all the H_{δ}^s norms of \mathbf{v} are bounded by the H_{δ}^0 norm. Thus, it follows that if a solution to $\mathcal{L}\mathbf{v} = 0$ exists, it must be smooth —*elliptic regularity*.

3.4 Existence of solutions to the approximate Killing vector equation

We are now in the position of analysing the existence of solutions to the approximate Killing equation (3). Our main tools will be the Fredholm alternative and integration by parts. We begin by considering some auxiliary results.

3.4.1 Auxiliary existence results

The following result relating solutions to the approximate Killing equations to solutions to the KID equations will be needed in our main result:

Lemma 4. *Let $(\mathcal{S}, h_{ij}, K_{ij})$ be a complete, smooth asymptotically Euclidean initial data set for the Einstein vacuum field equations with n asymptotic ends and satisfying the Decay Assumptions 1. Then, for $0 < \beta \leq 1/2$,*

$$\ker\{\mathcal{P} \circ \mathcal{P}^* : H_{\beta}^{\infty} \rightarrow H_{\beta-4}^{\infty}\} = \ker\{\mathcal{P}^* : H_{\beta}^{\infty} \rightarrow H_{\beta-2}^{\infty}\}$$

That is to say, the equation

$$\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} N \\ N^i \end{pmatrix} = 0$$

admits a solution $(N, N^i) \in H_{\beta}^{\infty}$, $0 < \beta \leq 1/2$, if and only if $(\mathcal{S}, h_{ij}, K_{ij})$ is stationary in the sense of Definition 2. Moreover, if the solution exists then it is unique up to constant rescaling.

Proof. Assume that $\mathcal{P} \circ \mathcal{P}^*(N, N^i) = 0$. Making use of the identity (4) with

$$\begin{pmatrix} \gamma^{ij} \\ q^{kij} \end{pmatrix} = \mathcal{P}^* \begin{pmatrix} N \\ N^i \end{pmatrix}$$

one finds that

$$\int_S \mathcal{P}^* \begin{pmatrix} N \\ N^i \end{pmatrix} \cdot \mathcal{P}^* \begin{pmatrix} N \\ N^i \end{pmatrix} d\mu = - \oint_{\partial S_{\infty}} n^k (\mathcal{A}_k + \mathcal{B}_k + \mathcal{C}_k + \mathcal{D}_k) dS$$

where $\partial\mathcal{S}_\infty$ denotes the sphere at infinity. We proceed now to evaluate the various boundary terms.

We observe that under the Decay Assumptions 1 direct inspection shows that

$$H_{ij} = o(|x|^{-2}),$$

from where it follows that

$$\gamma_{ij} = D_i D_j N - N r_{ij} - \Delta N h_{ij} + H_{ij} = o(|x|^{-3/2}).$$

Hence, one has that

$$\mathcal{A}_k = N D^i \gamma_{ik} - D^i N \gamma_{ik} + D_k N \gamma - N D_k \gamma = o(|x|^{-2}).$$

Thus, taking into account that $dS = O(|x|^2)$ one concludes that

$$\oint_{\partial\mathcal{S}_\infty} n^k \mathcal{A}_k dS = 0.$$

Next, we consider

$$\mathcal{C}_k = N^i D^l q_{lik} - D^j N^i q_{kij} + D_i N^i q_{kj}^j - N_i D^l q_{lj}^j$$

where

$$q_{kij} = D_k (D_{(i} N_{j)}) - D^l N_l h_{ij} - F_{ij}, \quad F_{ij} = 2N(Kh_{ij} - K_{ij}).$$

From the Decay Assumptions 1 it follows that in this case

$$F_{ij} = o(|x|^{-1}), \quad q_{kij} = o(|x|^{-3/2})$$

so that

$$\mathcal{C}_k = o(|x|^{-2}).$$

Thus, one has that

$$\oint_{\partial\mathcal{S}_\infty} \mathcal{C}_k n^k dS = 0.$$

Finally, similar considerations give that

$$\mathcal{D}_k = \frac{1}{2} N_k K^{lj} \gamma_{jl} + \frac{1}{2} N^i K_{ik} \gamma - N^i K_i^l \gamma_{kl} = o(|x|^{-5/2})$$

so that

$$\oint_{\partial\mathcal{S}_\infty} n^k \mathcal{D}_k dS = 0.$$

From the previous discussion it follows then that

$$\int_{\mathcal{S}} \mathcal{P}^* \begin{pmatrix} N \\ N^i \end{pmatrix} \cdot \mathcal{P}^* \begin{pmatrix} N \\ N^i \end{pmatrix} d\mu = 0$$

so that $\mathcal{P}^*(N, N^i) = 0$, and therefore that the data is stationary. Finally, uniqueness of the solution follows from Proposition 2. Suppose, for contradiction, that there exist two distinct solutions, giving rise to two distinct KID sets $(N, -\frac{1}{2}N^i)$ and $(\tilde{N}, -\frac{1}{2}\tilde{N}^i)$. Taking the appropriate linear combination we arrive at a KID set with a lapse that goes to zero at infinity while the shift is in H_β^∞ , $\beta \leq 1/2$ —that is, one has a KID associated to a spatial translation. This contradicts the fact that the ADM 4-momentum of the initial data is non-vanishing—see Section III in [5]. \square

Remark 9. Making use of the asymptotic expansion provided by Proposition 2 one finds that for stationary initial data sets, the solutions provided by Lemma 4 are of the form:

$$N - \mathfrak{A}^0 = o_\infty(|x|^{-1/2}), \quad N^\alpha - \mathfrak{A}^\alpha = o_\infty(|x|^{-1/2}) \quad (7)$$

with the components of a \mathfrak{A}^μ a constant vector field.

3.4.2 Main existence result

Following [13] we now will look for solutions of the approximate Killing equation such that

$$\begin{aligned} X &= \lambda|x| + \vartheta, & \vartheta &\in H_{1/2}^\infty, \\ X^i &\in H_{1/2}^\infty. \end{aligned}$$

in each asymptotically Euclidean end and where λ is a constant. This Ansatz is motivated by the observation that $\Delta_{\mathbf{g}}^2|x| = 0$, with $\Delta_{\mathbf{g}}$ the flat Laplacian—that is, the blowing up term $\lambda|x|$ is in the kernel of the first component of the operator $\mathcal{P} \circ \mathcal{P}^*$ evaluated on the 3-dimensional flat metric.

Theorem 1. *Let $(\mathcal{S}, h_{ij}, K_{ij})$ be a complete, smooth asymptotically Euclidean initial data set for the Einstein vacuum field equations with n asymptotic ends, satisfying the Decay Assumptions 1. Then there exists a solution (X, X^i) to the approximate KID equation,*

$$\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} X \\ X^i \end{pmatrix} = 0,$$

such that at each asymptotic end one has the asymptotic behaviour

$$\begin{aligned} X_{(k)} &= \lambda_{(k)}|x| + \vartheta_{(k)}, & \vartheta_{(k)} &\in H_{1/2}^\infty, \\ X_{(k)}^i &\in H_{1/2}^\infty, \end{aligned}$$

where $\lambda_{(k)}$, $k = 1, \dots, n$, are constants and $\lambda_{(k)} = 0$ for some k if and only if $(\mathcal{S}, h_{ij}, K_{ij})$ is stationary in the sense of Definition 2. Moreover, the solution is unique up to constant rescaling.

Proof. Substituting the above Ansatz in equation (3) one obtains

$$\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} \vartheta \\ X^i \end{pmatrix} = -\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} \lambda|x| \\ 0 \end{pmatrix}. \quad (8)$$

Under the Decay Assumptions 1, a lengthy computation shows that

$$\begin{aligned} H_{ij} &= o(|x|^{-2}), & F_{ij} &= o(|x|^{-1/2}), & Q_{ij} &= o(|x|^{-5/2}), & \bar{\gamma}_{ij} &= o(|x|^{-1}), \\ F_i &= o(|x|^{-7/2}), & H &= o(|x|^{-4}), \end{aligned}$$

where, in particular, it has been used that

$$\partial_\alpha|x| = \frac{x_\alpha}{|x|} = O(1), \quad \partial_\alpha\partial_\beta|x| = \frac{\delta_{\alpha\beta}}{|x|} - \frac{x_\alpha x_\beta}{|x|^3} = O(|x|^{-1}).$$

Hence,

$$\begin{aligned} 2\Delta_{\mathbf{h}}\Delta_{\mathbf{h}}X - r^{ij}D_iD_jX + 2r\Delta_{\mathbf{h}}X + D^iD^jH_{ij} - \Delta_{\mathbf{h}}H_k{}^k + \bar{H} \\ + \frac{3}{2}D^i r D_i X + (\frac{1}{2}\Delta_{\mathbf{h}}r + r_{ij}r^{ij})X - r^{ij}H_{ij} = o(|x|^{-7/2}), \\ D^j\Delta_{\mathbf{h}}F_{ij} - D_i\Delta_{\mathbf{h}}F_k{}^k - \bar{F}_i = o(|x|^{-7/2}). \end{aligned}$$

so that

$$\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} \lambda|x| \\ 0 \end{pmatrix} \in H_{-7/2}^0.$$

To prove the existence of solutions to equation (8) we make use of the Fredholm alternative in weighted Sobolev spaces, according to which equation (8) will have solution (ϑ, X^i) if and only if its right-hand-side is L^2 -orthogonal to $\text{coker}\{\mathcal{P} \circ \mathcal{P}^* : H_{1/2}^4 \rightarrow H_{1/2}^0\}$ —i.e. if and only if

$$\int_{\mathcal{S}} \mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} \lambda|x| \\ 0 \end{pmatrix} \cdot \begin{pmatrix} N \\ N^i \end{pmatrix} d\mu = 0$$

for all $(N, N^i) \in H_{1/2}^0$ for which

$$\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} N \\ N^i \end{pmatrix} = 0.$$

From Lemma 4 we know that this equation has non-trivial solutions (i.e. that $\text{coker}\{\mathcal{P} \circ \mathcal{P}^* : H_{1/2}^4 \rightarrow H_{1/2}^0\}$ will be non-trivial) if and only if $(\mathcal{S}, h_{ij}, K_{ij})$ is stationary. Thus, if the initial data set is not stationary, the Fredholm alternative guarantees a solution (ϑ, X^i) to (8).

For the stationary case, the cokernel is spanned by a single Killing vector with components (N, N_i) , taking the form of (7). Let

$$\begin{pmatrix} \Gamma_{ij} \\ Q_{kij} \end{pmatrix} \equiv \mathcal{P}^* \begin{pmatrix} \lambda|x| \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda(D_i D_j |x| - |x| r_{ij} - \Delta |x| h_{ij}) + H_{ij} \\ D_k F_{ij} \end{pmatrix}$$

where, now,

$$\begin{aligned} H_{ij} &\equiv 2\lambda|x|(K^k{}_i K_{jk} - K K_{ij}) = o(|x|^{-2}), \\ F_{ij} &\equiv 2\lambda|x|(K h_{ij} - K_{ij}) = o(|x|^{-1/2}). \end{aligned}$$

It then follows that

$$\Gamma_{ij} = o(|x|^{-1}), \quad Q_{kij} = o(|x|^{-3/2})$$

and that

$$\mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} \lambda|x| \\ 0 \end{pmatrix} = \mathcal{P} \begin{pmatrix} \Gamma_{ij} \\ Q_{kij} \end{pmatrix} = o(|x|^{-7/2})$$

Then, using the identity (4) and the fact that, by assumption, $\mathcal{P}^*(N, N^i) = 0$, we see that

$$\int_S \mathcal{P} \circ \mathcal{P}^* \begin{pmatrix} \lambda|x| \\ 0 \end{pmatrix} \cdot \begin{pmatrix} N \\ N_i \end{pmatrix} d\mu = \oint_{\partial S_\infty} n^k (\mathcal{A}_k + \mathcal{B}_k + \mathcal{C}_k + \mathcal{D}_k) dS \quad (9)$$

where, here

$$\begin{aligned} \mathcal{A}_k &\equiv N D^j \Gamma_{jk} - D^j N \Gamma_{jk} - D_k N \Gamma - N D_k \Gamma, \\ \mathcal{B}_k &\equiv 2(K^{ij} Q_{kij} - K Q_{kj}{}^j) N, \\ \mathcal{C}_k &\equiv N^i D^l Q_{lik} - D^j N^i Q_{kij} + D_i N^i Q_{kj}{}^j - N_i D^l Q_{lj}{}^j, \\ \mathcal{D}_k &\equiv N^i K_i{}^l \Gamma_{kl} - \frac{1}{2} N_k K^{lj} \Gamma_{jl} - \frac{1}{2} N^i K_{ik} \Gamma. \end{aligned}$$

and $\Gamma \equiv h^{ij} \Gamma_{ij}$. We find then that

$$\mathcal{B}_k = o(|x|^{-3}), \quad \mathcal{C}_k = o(|x|^{-5/2}), \quad \mathcal{D}_k = o(|x|^{-5/2})$$

and

$$\mathcal{A}_k = -4\lambda \mathfrak{A}^0 |x|^{-2} n_k + o(|x|^{-5/2})$$

Therefore, the only contribution to the right-hand-side of (9) is the following

$$\oint_{\partial S_\infty} n^k \mathcal{A}_k dS = -4\lambda \mathfrak{A}^0 \oint_{\partial S_\infty} |x|^{-2} dS = -16\pi \lambda \mathfrak{A}^0$$

Since $\mathfrak{A}^0 \neq 0$, we see that in the stationary case we have an obstruction to solving (8), unless $\lambda = 0$, in which case $(\vartheta, X_i) = (N, N_i)$ is the unique solution, up to constant rescaling.

Finally, in the non-stationary case, uniqueness of the solution (X, X^i) follows by an argument analogous to that of Lemma 4. Suppose, for contradiction, that we have two linearly-independent solutions $(\lambda|x| + \vartheta, X^i)$, $(\tilde{\lambda}|x| + \tilde{\vartheta}, \tilde{X}^i)$. Since the initial data is by assumption non-stationary, then $\lambda \neq 0$ and $\tilde{\lambda} \neq 0$. Hence, taking the appropriate linear combination we arrive at a non-trivial approximate KID with lapse and shift in $H_{1/2}^\infty$. This contradicts the conclusions of Lemma 4. Hence we conclude that $(\lambda|x| + \vartheta, X^i)$ and $(\tilde{\lambda}|x| + \tilde{\vartheta}, \tilde{X}^i)$ are linearly dependent —i.e. the solution is unique up to constant rescaling.

□

Remark 10. The fact that $(\vartheta, N^i) \in H_{1/2}^\infty$ in the previous theorem follows from an application of Proposition 4 to equation (8).

Remark 11. In [13], the invariant λ is also given as a bulk integral as follows

$$\lambda = \frac{1}{16\pi} \int_S X r_{ij} r^{ij} d\mu$$

The above integral formula, valid only in the time-symmetric case, is derived from the following boundary integral

$$\lambda = -\frac{1}{8\pi} \oint_{\partial S_\infty} n^k D_k \Delta(\lambda|x|) dS,$$

through the use of the divergence theorem, substitution using the approximate KID equation and integration-by-parts. A similar calculation yields a more general formula, valid for all data sets satisfying Decay Assumption 1, in terms of both the lapse and the shift of the approximate KID set. The expression is however rather complicated, and so in the interest of conciseness it is not presented here. Nevertheless, it reduces to the above formula when time-symmetry is assumed. Further study of the integral formula is deferred to subsequent work.

4 The geometric invariant in conformally flat initial data sets

We have seen in the previous section that an approximate Killing vector with lapse of the form $\eta = \lambda|x| + \vartheta$ exists for general asymptotically flat data, and moreover, that the constant λ vanishes if and only if the spacetime development is stationary. In this section we analyse further the asymptotic properties of the solutions to the approximate Killing vector equation in the case of conformally flat initial data sets.

4.1 Solutions to the Poisson equation in \mathbb{R}^3

We start with some mathematical preliminaries. Let us assume for the remainder of this section that $K_{ij} = O(|x|^{-3+\epsilon})$, for any $\epsilon > 0$. It follows then from the Hamiltonian constraint that

$$r = -K^2 + K_{ij}K^{ij} = O(|x|^{-6+2\epsilon}).$$

Moreover, the lapse component of the approximate Killing vector equation can be found to satisfy

$$2\Delta_h \Delta_h \eta - r^{ij} D_i D_j \eta + r_{ij} r^{ij} \eta = O(|x|^{-11/2+\epsilon}). \quad (10)$$

As is well known, the harmonic functions on \mathbb{R}^3 are spanned by functions of the forms

$$Q_{\alpha_1 \dots \alpha_k} x^{\alpha_1} \dots x^{\alpha_k}, \quad \frac{Q_{\alpha_1 \dots \alpha_k} x^{\alpha_1} \dots x^{\alpha_k}}{|x|^{2k+1}}, \quad k = 0, 1, 2, \dots,$$

where $Q_{\alpha_1 \dots \alpha_k}$ are symmetric trace-free tensors with constant coefficients. The following result will prove useful:

Lemma 5 (Meyers, [18]). *Let δ denote the flat 3-metric and $G = O(|x|^{-2-p-\epsilon}(\ln|x|)^q)$ a Hölder continuous function. Then the equation*

$$\Delta_\delta V = G \quad (11)$$

admits a solution V^\star satisfying

$$V^\star(x) = \begin{cases} O(|x|^{-p-\epsilon}(\ln|x|)^q) & \text{if } 0 < p < 1 \text{ or } \epsilon > 0, \\ O(|x|^{-p}(\ln|x|)^{q+1}) & \text{otherwise.} \end{cases}$$

Remark 12. By linearity of the Poisson equation (8), any two solutions thereof differ only by harmonic terms. In particular, the most general solution $V(x)$ of (11), assuming $V = O(|x|^r)$ for $r > -p$, is given by

$$V(x) = \begin{cases} V^*(x) + \sum_{k=\lfloor -r \rfloor}^{\lfloor p-1 \rfloor} Q_{\alpha_1 \dots \alpha_k} \frac{x^{\alpha_1} \dots x^{\alpha_k}}{|x|^{2k+1}} & \text{if } r < 0, \\ V^*(x) + \sum_{k=0}^{\lfloor p-1 \rfloor} Q_{\alpha_1 \dots \alpha_k} \frac{x^{\alpha_1} \dots x^{\alpha_k}}{|x|^{2k+1}} + \sum_{l=0}^{\lfloor r \rfloor} \widehat{Q}_{\alpha_1 \dots \alpha_l} x^{\alpha_1} \dots x^{\alpha_l} & \text{if } r > 0. \end{cases}$$

for some symmetric, trace-free $Q_{\alpha_1 \dots \alpha_k}$, $\widehat{Q}_{\alpha_1 \dots \alpha_l}$ with constant coefficients and where for a real number p , $\lfloor p \rfloor$ denotes the floor of p —i.e. the largest integer smaller than p . It will be useful to note that, for $k \in \mathbb{Z}$,

$$\Delta_{\delta} \left(\frac{x^{\alpha}}{|x|^k} \right) = k(k-3) \frac{x^{\alpha}}{|x|^{k+2}}.$$

4.2 Conformally flat initial data sets

We consider now maximal conformally-flat data initial data sets, i.e. collections $(\mathcal{S}, h_{ij}, K_{ij})$ such that

$$h_{ij} = \phi^4 \delta_{ij}, \quad K_{ij} = P_{ij}$$

where $\phi \rightarrow 1$ as $|x| \rightarrow \infty$ and $P_{\alpha\beta} = O(|x|^{-3+\epsilon})$ is a symmetric, tracefree and divergence free with respect to the flat metric. It will also prove convenient to define $\psi_{\alpha\beta} \equiv \phi^2 P_{\alpha\beta}$, in terms of which the Hamiltonian and momentum constraints take the familiar forms

$$\Delta_{\delta} \phi = -\frac{1}{8} \phi^{-7} \psi_{\alpha\beta} \psi^{\alpha\beta}, \quad (12a)$$

$$\partial^{\alpha} \psi_{\alpha\beta} = 0, \quad (12b)$$

where indices are now raised and lowered with respect to the flat metric, δ_{ij} . Then, it follows from (12a) and an application of Lemma 5 that

$$\phi = 1 + \frac{2m}{|x|} + \frac{L_{\alpha} x^{\alpha}}{|x|^3} + \frac{A_{\alpha\beta} x^{\alpha} x^{\beta}}{|x|^5} + O\left(\frac{\ln |x|}{|x|^{4-2\epsilon}}\right) \quad (13)$$

for some constant m , and constant-coefficient L_{α} , $A_{\alpha\beta}$, which are independent of the extrinsic curvature $P_{\alpha\beta}$ which contributes only at order $O(\ln |x|/|x|^{4-2\epsilon})$.

In terms of the flat connection, equation (10) becomes

$$\begin{aligned} \Delta_{\delta} \Delta_{\delta} \eta + A(\phi)^{\alpha} \partial_{\alpha} \Delta_{\delta} \eta + B(\phi)^{\alpha\beta} \partial_{\alpha} \partial_{\beta} \eta + B(\phi) \Delta_{\delta} \eta \\ + C(\phi)^{\alpha} \partial_{\alpha} \eta + D(\phi) \eta = O(|x|^{-11/2+\epsilon}) \end{aligned} \quad (14)$$

where

$$\begin{aligned} A(\phi)^{\alpha} &\equiv -4\phi^{-1} \partial^{\alpha} \phi, \\ B(\phi)^{\alpha\beta} &\equiv \phi^{-2} (5\phi \partial^{\alpha} \partial^{\beta} \phi - 19 \partial^{\alpha} \phi \partial^{\beta} \phi), \\ B(\phi) &\equiv 13\phi^{-2} |\partial \phi|^2, \\ C(\phi)^{\alpha} &\equiv 4\phi^{-3} (12 |\partial \phi|^2 \partial^{\alpha} \phi - 5\phi (\partial_{\beta} \phi) \partial^{\beta} \partial^{\alpha} \phi), \\ D(\phi) &\equiv 2\phi^{-4} (6 |\partial \phi|^4 - 6\phi (\partial^{\alpha} \phi) (\partial^{\beta} \phi) \partial_{\alpha} \partial_{\beta} \phi + \phi^2 (\partial^{\alpha} \partial^{\beta} \phi) (\partial_{\alpha} \partial_{\beta} \phi)). \end{aligned}$$

Proposition 5. *Let $(\mathcal{S}, h_{ij}, P_{ij})$ be maximal conformally-flat data with $P_{\alpha\beta} = O(|x|^{-3+\epsilon})$, then the lapse of the approximate Killing vector has an asymptotic expansion of the form*

$$\begin{aligned} \eta = \lambda |x| + 18\lambda m \ln |x| + Q_{\alpha} \frac{x^{\alpha}}{|x|} + Q^{(1)} - 104\lambda m^2 \frac{\ln |x|}{|x|} \\ + \frac{Q^{(2)}}{|x|} - \frac{1}{4} (23\lambda L_{\alpha} - 26m Q_{\alpha}) \frac{x^{\alpha}}{|x|^2} + Q_{\alpha\beta} \frac{x^{\alpha} x^{\beta}}{|x|^3} + O(|x|^{-3/2}) \end{aligned} \quad (15)$$

for some constants $Q^{(1)}, Q^{(2)}, Q_{\alpha}, Q_{\alpha\beta}$, where m and L_{α} are the constants appearing in (13).

Proof. Substituting (13) into (14) one obtains

$$\begin{aligned} \Delta_\delta \Delta_\delta \eta + \frac{4}{|x|^3} \left[2mx^\alpha - 4m^2 \frac{x^\alpha}{|x|} - L^\alpha + 3L_\beta \frac{x^\beta x^\alpha}{|x|^2} \right] \partial_\alpha \Delta_\delta \eta \\ + \frac{15}{|x|^5} \left[\left(2m - \frac{4m^2}{|x|} + \frac{5}{|x|^2} L_\gamma x^\gamma \right) x^\alpha x^\beta - 2L^{(\alpha} x^{\beta)} \right] \partial_\alpha \partial_\beta \eta \\ - \frac{1}{|x|^3} \left[10m - \frac{72m^2}{|x|} + \frac{15}{|x|^2} L_\gamma x^\gamma \right] \Delta_\delta \eta + \frac{160m^2}{|x|^6} x^\alpha \partial_\alpha \eta + \frac{48m^2}{|x|^6} \eta = O(|x|^{-11/2+\epsilon}). \end{aligned} \quad (16)$$

Substituting the Ansatz, $\eta = \lambda|x| + \vartheta$, where $\vartheta = o(|x|^{1/2})$, and collecting lower-order terms in ϑ

$$\Delta_\delta \Delta_\delta \vartheta = \frac{36\lambda m}{|x|^4} + O(|x|^{-9/2}).$$

Using Lemma 5, we obtain

$$\Delta_\delta \vartheta = \frac{18\lambda m}{|x|^2} - 2Q_\alpha \frac{x^\alpha}{|x|^3} + O(|x|^{-5/2})$$

for some constant-coefficient Q_α . Here we have used that $\Delta_\delta \vartheta = o(|x|^{-3/2})$, thereby excluding constant and $1/|x|$ harmonic terms. Applying Lemma 5 again we find that

$$\vartheta = 18\lambda m \ln |x| + Q_\alpha \frac{x^\alpha}{|x|} + Q^{(1)} + \varphi$$

for some constant $Q^{(1)}$, and some function $\varphi = O(|x|^{-1/2})$. Substituting into (16),

$$\Delta_\delta \Delta_\delta \varphi = \frac{624\lambda m^2}{|x|^5} + (46\lambda L_\alpha - 52mQ_\alpha) \frac{x^\alpha}{|x|^6} + O(|x|^{-11/2+\epsilon})$$

implying that

$$\Delta_\delta \varphi = \frac{104\lambda m^2}{|x|^3} + \frac{1}{2}(23\lambda L_\alpha - 26mQ_\alpha) \frac{x^\alpha}{|x|^4} + Q_{\alpha\beta} \frac{x^\alpha x^\beta}{|x|^5} + O(|x|^{-7/2+\epsilon})$$

for some constant-coefficient, tracefree $Q_{\alpha\beta}$. Here we have used the fact that $\Delta_\delta \varphi = O(|x|^{-5/2})$ to eliminate constant, $1/|x|$ and $1/|x|^2$ harmonic terms. Integrating up once more, we obtain (15). \square

It is interesting to note the presence of a logarithmically-singular term in (15) in the non-Killing case, $\lambda \neq 0$. On the other hand, if one sets $\lambda = 0$ in (15), then one obtains the asymptotic expansion

$$\eta = Q_\alpha \frac{x^\alpha}{|x|} + Q^{(1)} + \frac{Q^{(2)}}{|x|} + \frac{13}{2}mQ_\alpha \frac{x^\alpha}{|x|^2} + Q_{\alpha\beta} \frac{x^\alpha x^\beta}{|x|^3} + O(|x|^{-3/2}) \quad (17)$$

for the lapse of a general spacetime Killing vector restricted to a conformally flat initial data set.

Conclusions

We have shown that the existence of approximate Killing vectors extends to a large class of asymptotically Euclidean initial data with non-vanishing extrinsic curvature and non-vanishing ADM 4-momentum. Following Dain's discussion in [13], we can then define a geometric invariant $\lambda_{(k)}$ on each asymptotically Euclidean end given by the leading coefficient in the appropriate asymptotic expansion. The vanishing of any one of the $\lambda_{(k)}$ characterises stationarity of the initial data.

Further work would involve the construction of approximate Killing vectors on hyperboloidal hypersurfaces. It would also be of interest to explore the dynamics of the approximate KID. If a propagation equation for the approximate KID can be found, one may be able to use Dain's invariant to quantify the deviation from stationarity of a generic asymptotically Euclidean initial data set.

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